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CONJUGATE CONVEX FUNCTIONS IN TOPOLOGICAL VECTOR SPACES

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Synopsis

Continuing investigations by W. L. Jones (Thesis, Columbia University, 1960), the theory of conjugate convex functions in finite-dimensional Euclidean spaces, as developed by W. Fenchel (Canadian J. Math. 1 (1949) and Lecture Notes, Princeton University, 1953), is generalized to functions in locally convex topological vector spaces.

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Introduction

The purpose of the present paper is to generalize the theory of conjugate convex functions in finite-dimensional Euclidean spaces, as initiated by Z. Birnbaum and W. Orlicz [1] and S. Mandelbrojt [8] and developed by W. Fenchel [3], [4] (cf. also S. Karlin [6]), to infinite-dimensional spaces. To a certain extent this has been done previously by W. L. Jones in his Thesis [5]. His principal results concerning the conjugates of real functions in topological vector spaces have been included here with some improvements and simplified proofs (Section 3). After the present paper had been written, the author's attention was called to papers by J. J. Moreau [9], [10], [11] in which, by a different approach and independently of Jones, results equivalent to many of those contained in this paper (Sections 3 and 4) are obtained.

Section 1 contains a summary, based on [7], of notions and results from the theory of topological vector spaces applied in the following. Section 2 deals with real functions f defined on subsets D of a locally convex topological vector space. In particular convex functions are considered. In Sections 3 and 4 the theory of conjugate functions is developed. The starting point is a pair of locally convex topological vector spaces E_1, E_2 which are (topological) duals of each other. For a function f with domain $D \subseteqq E_1$, briefly denoted by (D, f), we define

$$D' = \left\{ \xi \in E_2 \,\middle|\, \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) < \infty \right\}.$$

If D' is non-empty, the function (D', f'), where

$$f'(\xi) = \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) \qquad \text{for } \xi \in D',$$

is called the conjugate of (D, f). Analogously the conjugate of a function in E_2, in particular the second conjugate (D'', f'') of (D, f), that is, the conjugate of (D', f'), is defined. In Section 3 elementary properties of (D', f') and (D'', f'') are studied, and necessary and sufficient conditions in order that a convex function (D, f) have a conjugate and that it coincide with its second conjugate are given. In Section 4 the conjugates of functions derived from others in various ways are determined. Finally, in Section 5 the class of convex functions (D, f) in E_1 which coincide with their second conjugates and have the property that their domains D as well as the domains D' of their conjugates have relative interior points is characterized in the case of Banach spaces E_1 and E_2.

1. Topological vector spaces

In the following R denotes the set of reals, R_+ the set of positive reals and Z_+ the set of positive intergers. When R is considered as a topological space, the topology is the usual one. All the vector spaces considered are vector spaces over R.

Let E be a vector space over R with elements $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{o}$ being the zero element, and let $\mathfrak{T}$ be a Hausdorff topology on E. If the mappings $E \times E \rightarrow E$ defined by $(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \boldsymbol{x}+\boldsymbol{y}$ and $R \times E \rightarrow E$ defined by $(a, \boldsymbol{x}) \rightarrow a\boldsymbol{x}$ are continuous, E is said to be a topological vector space. $\mathfrak{T}$ is then called a vector space topology, and E is denoted $E[\mathfrak{T}]$.

If F is an algebraic subspace of $E[\mathfrak{T}]$, then F is a topological vector space in the induced topology.

If $\mathfrak{B} = \{V\}$ is a basis for the neighbourhood system of $\boldsymbol{o}$ in $E[\mathfrak{T}]$, then $\{\boldsymbol{x}+V \mid V \in \mathfrak{B}\}$ is a basis for the neighbourhood system of the point $\boldsymbol{x}$. There exists a basis $\mathfrak{B}$ such that all $V \in \mathfrak{B}$ are symmetric, in the sense that $\boldsymbol{x} \in V$ and $|a| \leq 1$ imply $a\boldsymbol{x} \in V$. If $b \neq 0$ and $V \in \mathfrak{B}$, then $bV \in \mathfrak{B}$. A subset M of $E[\mathfrak{T}]$ is said to be bounded if for all $V \in \mathfrak{B}$ there exists a $c \in R$ such that $M \subseteqq cV$. Any set consisting of only one point is bounded.

The dual space of $E[\mathfrak{T}]$, denoted $(E[\mathfrak{T}])'$, is the set of continuous linear functions on $E[\mathfrak{T}]$, organized as a vector space in the well-known manner. For $\xi \in (E[\mathfrak{T}])'$ the value of ξ at $\boldsymbol{x} \in E[\mathfrak{T}]$ is denoted $\xi\boldsymbol{x} = \boldsymbol{x}\xi$.

A set $M \subseteqq E[\mathfrak{T}]$ is called convex if $(1-t)\boldsymbol{x}+t\boldsymbol{y} \in M$ for all $\boldsymbol{x}, \boldsymbol{y} \in M$, $0 \leq t \leq 1$. The set $a_1M_1+a_2M_2+\ldots+a_nM_n$ is convex for M_i convex, $a_i \in R$, $i = 1, 2, \ldots n$. The intersection of convex sets is convex. The smallest convex set containing a set M, i.e. the intersection of all convex sets containing M, is denoted $\operatorname{conv} M$. It consists of all points $\boldsymbol{x} = \sum_{i=1}^{i=n} a_i\boldsymbol{x}_i$, where $a_i \in R$, $\boldsymbol{x}_i \in M$, $n \in Z_+$ and $\sum_{i=1}^{i=n} a_i = 1$. The set $\overline{\operatorname{conv} M}$ is the intersection of all closed convex sets containing M, and $\overline{\operatorname{conv} M} = \overline{\operatorname{conv} \overline{M}}$. If M is convex, then also the closure $\overline{M}$ of M and the interior $\mathring{M}$ of M are convex sets. If $\mathring{M}$ is not empty, then $(1-t)\boldsymbol{x}+t\boldsymbol{y} \in \mathring{M}$ for all $\boldsymbol{y} \in \mathring{M}$, $\boldsymbol{x} \in \overline{M}$ and $0 < t \leq 1$. We express this by saying that all points in $\overline{M}$ can be reached from $\mathring{M}$. Further $\mathring{M} = \mathring{\overline{M}}$ and $\overline{M} = \overline{\mathring{M}}$.

A topological vector space is said to be locally convex if there exists a basis for the neighbourhood system of $\boldsymbol{o}$ consisting of convex sets. Normed spaces are locally convex topological vector spaces. A subspace of a locally convex space is itself locally convex.

Let E_1 and E_2 be vector spaces over R, and suppose that there exists a bilinear mapping $\mathfrak{B} : E_2 \times E_1 \to R$ with the following two properties:

(i) For all $\boldsymbol{x} \in E_1$, $\boldsymbol{x} \neq \boldsymbol{o}$, there exists a $\xi \in E_2$ such that $\mathfrak{B}(\xi, \boldsymbol{x}) \neq 0$.

(ii) For all $\xi \in E_2$, $\xi \neq \boldsymbol{o}$, there exists an $\boldsymbol{x} \in E_1$ such that $\mathfrak{B}(\xi, \boldsymbol{x}) \neq 0$.

We then say that E_1 and E_2 are in duality under $\mathfrak{B}$.

Let E_1 and E_2 be in duality under $\mathfrak{B}$. Then every $\xi \in E_2$ is a linear function on E_1, and every $\boldsymbol{x} \in E_1$ is a linear function on E_2, namely $\xi\boldsymbol{x} = \boldsymbol{x}\xi = \mathfrak{B}(\xi, \boldsymbol{x})$. There exist at least one locally convex topology $\mathfrak{T}_1$ on E_1 and at least one locally convex topology $\mathfrak{T}_2$ on E_2 such that E_2 is the set of continuous linear functions on $E_1[\mathfrak{T}_1]$ and E_1 is the set of continuous linear functions on $E_2[\mathfrak{T}_2]$, that is, $(E_1[\mathfrak{T}_1])' = E_2$ and $(E_2[\mathfrak{T}_2])' = E_1$. Such topologies are called admissible.

If $E[\mathfrak{T}]$ is locally convex, and E' is the dual space, then E and E' are in duality under $\mathfrak{B}(\xi, \boldsymbol{x}) = \xi\boldsymbol{x}$, and $\mathfrak{T}$ is an admissible topology on E.

Let $E[\mathfrak{T}]$ be a normed vector space, $\mathfrak{T}$ denoting the topology induced by the norm, and let E' be the dual space. It is well-known that $\| \xi \| = \sup_{\|\boldsymbol{x}\| \leq 1} | \xi\boldsymbol{x} |$ is a norm in E'. The topology $\mathfrak{T}'$ induced by this norm is admissible if and only if $E[\mathfrak{T}]$ is a reflexive Banach space. In that case $E'[\mathfrak{T}']$ is also a reflexive Banach space.

Let E_1 and E_2 be in duality under $\mathfrak{B}$. Defining $a(\boldsymbol{x}, x) = (a\boldsymbol{x}, ax)$ and $(\boldsymbol{x}, x) + (\boldsymbol{y}, y) = (\boldsymbol{x}+\boldsymbol{y}, x+y)$, $E_1 \times R$ is a vector space over R. Likewise for $E_2 \times R$. Further, $E_1 \times R$ and $E_2 \times R$ are in duality under $\tilde{\mathfrak{B}}((\xi, \xi), (\boldsymbol{x}, x)) = \mathfrak{B}(\xi, \boldsymbol{x}) + \xi x$. If $\mathfrak{T}_1$ is a locally convex topology on E_1, then $E_1[\mathfrak{T}_1] \times R$, that is, $E_1 \times R$ supplied with the product space topology, is a locally convex space. If $\mathfrak{T}_1$ is admissible, then the topology on $E_1[\mathfrak{T}_1] \times R$ is also admissible. A basis for the neighbourhood system of the zero element $(\boldsymbol{o}, 0)$ in $E_1[\mathfrak{T}_1] \times R$ consists of all sets of the form $V \times R_\varepsilon$, where V is in a basis for the neighbourhood system of $\boldsymbol{o}$ in $E_1[\mathfrak{T}_1]$, $\varepsilon \in R_+$ and $R_\varepsilon = \{ a \in R \mid |a| \leq \varepsilon \}$. Likewise for E_2.

Let E_1 and E_2 be in duality. For a subset M of E_1 we define

$$M^\perp = \{ \xi \in E_2 \mid \xi\boldsymbol{x} = 0 \text{ for all } \boldsymbol{x} \in M \}.$$

Likewise for $M \subseteq E_2$. Putting $(M^\perp)^\perp = M^{\perp\perp}$ and $(M^{\perp\perp})^\perp = M^{\perp\perp\perp}$, we have

$M \subseteq M^{\perp\perp}$ and $M^{\perp} = M^{\perp\perp\perp}$ for any subset M. Further, $M^{\perp}$ and $M^{\perp\perp}$ are subspaces which are closed in every admissible topology, and $M = M^{\perp\perp}$ if and only if M is a subspace which is closed in every admissible topology.

A linear manifold in a vector space E is a set of the form $\boldsymbol{y}+H$, where H is a subspace in E. The intersection of closed linear manifolds is a closed linear manifold. The intersection of all closed linear manifolds containing a subset M of $E[\mathfrak{T}]$ is denoted $m(M)$. An interior point of M in $m(M)$ is called a relative interior point of M.

If $\boldsymbol{y}+H$ is a linear manifold in E, and H has codimension one, then $\boldsymbol{y}+H$ is called a hyperplane. For every hyperplane $\boldsymbol{y}+H$ in E there exists a linear function ξ on E and a $c \in R$ such that $\boldsymbol{y}+H = \{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} = c\}$. Conversely, if ξ is a linear function on E and c is in R, then the set $\{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} = c\}$ is a hyperplane in E. Let $E[\mathfrak{T}]$ be a topological vector space. Then the hyperplane $\boldsymbol{y}+H$ is closed if and only if a corresponding linear function ξ is continuous. A non-closed hyperplane is dense in $E[\mathfrak{T}]$.

For $\xi \in (E[\mathfrak{T}])'$ and $c \in R$, the sets $\{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} \leq c\}$ and $\{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} \geq c\}$ are called the closed halfspaces determined by the hyperplane. They are closed convex sets. The sets $\{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} < c\}$ and $\{\boldsymbol{x} \in E \mid \xi\boldsymbol{x} > c\}$ are called the open halfspaces determined by the hyperplane. They are open convex sets. A closed hyperplane in $E[\mathfrak{T}]$ is said to separate the sets A and B if A is contained in one of the two closed halfspaces determined by the hyperplane and B is contained in the other one. If A is contained in one of the open halfspaces and B in the other one, then the hyperplane is said to separate strictly. In that case A and B have no common points.

In locally convex spaces we have the following theorems:

1.1 If A is a closed convex set and $\boldsymbol{x}$ is a point not contained in A, then there exists a closed hyperplane which separates A and $\{\boldsymbol{x}\}$ strictly ([7] p. 245).

1.2. Every closed convex set A is the intersection of all closed halfspaces containing it ([7] p. 246).

2. Functions (D, f)

Let $E[\mathfrak{T}]$ be a locally convex topological vector space. Then also $E[\mathfrak{T}] \times R$ is a locally convex topological vector space.

The functions considered in the following are real functions defined on non-empty subsets of E. A pair consisting of a function f and its domain D will be denoted (D, f) and called a function in E.

If $\boldsymbol{x} \in D$ has a neighbourhood $\boldsymbol{x}+V$ such that $(\boldsymbol{x}+V) \cap (D \backslash \{\boldsymbol{x}\})$ is empty, then $\boldsymbol{x}$ is called an isolated point of D. Convex sets consisting of more than one point have no isolated points.

If (D, f) is a function in E, we define for a non-isolated point $\boldsymbol{x} \in \bar{D}$

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) = \sup_{V \in \mathfrak{V}} \left\{ \inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x}+V) \cap (D \backslash \{\boldsymbol{x}\}) \right\} \right\},$$

where $\mathfrak{V}$ is a basis for the neighbourhood system of $\boldsymbol{o}$ in E. If $\boldsymbol{x} \in D$ is isolated, we put

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) = f(\boldsymbol{x}).$$

Let (D, f) be a function in E. We say that f is *lower semi-continuous* at a point $\boldsymbol{x} \in D$ if

$$f(\boldsymbol{x}) \leqq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}).$$

If f is lower semi-continuous at every point of D, (D, f) is said to be lower semi-continuous.

If (D, f) is a lower semi-continuous function such that for all $\boldsymbol{x} \in \bar{D}$

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) < \infty \qquad \text{implies} \qquad \boldsymbol{x} \in D,$$

then (D, f) is said to be *closed*. Theorem 2.2 below motivates this definition.

A function (D, f) is said to be *convex* if D is a convex set and

$$f((1-t)\boldsymbol{x} + t\boldsymbol{y}) \leqq (1-t) f(\boldsymbol{x}) + t f(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in D$ and $0 \leqq t \leqq 1$.

For any function (D, f) we define

$$[D, f] = \left\{ (\boldsymbol{x}, x) \in E \times R \mid \boldsymbol{x} \in D, x \geqq f(\boldsymbol{x}) \right\}.$$

Then the following statement obviously holds.

2.1. *A function (D, f) is convex if and only if $[D, f]$ is a convex set.*

Further, we have

2.2. *A function (D, f) is closed if and only if $[D, f]$ is a closed set.*

Proof. For $a \in R$ we define

$$T_a = \left\{ \boldsymbol{x} \in D \mid f(\boldsymbol{x}) \leqq a \right\}.$$

If (D, f) is a closed function, then T_a is closed for every $a \in R$. For, $\boldsymbol{x}$ not in T_a implies $\boldsymbol{x}$ not in $\bar{D}$ or

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) > a.$$

Thus there exists a $V \in \mathfrak{V}$ such that $(\boldsymbol{x}+V) \cap T_a$ is empty, which proves that the complement of T_a is open. Now let $(\boldsymbol{y}, b) \in \overline{[D, f]}$. Then $\boldsymbol{y} \in \overline{T}_a = T_a$ for every $a > b$, whence $\boldsymbol{y} \in D$ and $f(\boldsymbol{y}) \leq b$. Consequently $(\boldsymbol{y}, b) \in [D, f]$, and so $[D, f]$ is closed.

Conversely, let $[D, f]$ be closed. Obviously the set

$$\{(\boldsymbol{x}, a) \in E \times R \mid \boldsymbol{x} \in E\}$$

is closed for every $a \in R$. Thus,

$$[D, f] \cap \{(\boldsymbol{x}, a) \mid \boldsymbol{x} \in E\}$$

is closed for every $a \in R$. This implies that T_a is closed for every $a \in R$. To prove that (D, f) is closed we consider an $\boldsymbol{x} \in \overline{D}$ such that

$$b = \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) < \infty .$$

Now $b = -\infty$ would imply $\boldsymbol{x} \in \overline{T}_a = T_a$ for every $a \in R$ which is impossible. Hence $b \in R$, and clearly $\boldsymbol{x} \in \overline{T}_b = T_b$. So $\boldsymbol{x} \in D$ and $f(\boldsymbol{x}) \leq b$ which proves that (D, f) is closed.

As a consequence of 1.2, 2.1 and 2.2 we have

2.3. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are locally convex vector space topologies on E such that $(E[\mathfrak{T}_1])' = (E[\mathfrak{T}_2])'$, then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ determine the same closed convex functions in E.*

2.4. *If (D, f) is a convex function, then $f(\boldsymbol{x}) \geq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z})$ for all $\boldsymbol{x} \in D$.*

Proof. If $D = \{\boldsymbol{x}\}$, there is nothing to prove. Otherwise consider a $\boldsymbol{y} \in D, \boldsymbol{y} \neq \boldsymbol{x}$. For $0 < t \leq 1$ we have $(1-t)\boldsymbol{x} + t\boldsymbol{y} \in D \setminus \{\boldsymbol{x}\}$. Then

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) \leq \liminf_{t \to 0} ((1-t)f(\boldsymbol{x}) + tf(\boldsymbol{y})) = f(\boldsymbol{x}).$$

From 2.4 we deduce

2.5. *A convex function (D, f) is closed if and only if*

$$D = \{\boldsymbol{x} \in \overline{D} \mid \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) < \infty\}$$

and

$$f(\boldsymbol{x}) = \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) \qquad \textit{for } \boldsymbol{x} \in D.$$

The following four propositions are valid for arbitrary topological vector spaces.

2.6. *If (D, f) is a convex function, and there exists an $\boldsymbol{x} \in \bar{D}$ such that f is bounded below in some neighbourhood of $\boldsymbol{x}$ in D, then we have:*

(i) *f is bounded below on every bounded subset of D.*

(ii) *Every point of $\bar{D}$ has a neighbourhood in D in which f is bounded below.*

Proof. We may assume that $\boldsymbol{x} = \boldsymbol{o} \in D$ and $f(\boldsymbol{o}) = 0$. Let $b \in R$ and $V \in \mathfrak{V}$, V open and symmetric, be such that $f(\boldsymbol{z}) \geqq b$ for all $\boldsymbol{z} \in V \cap D$. Let $0 < t \leqq 1$. For every $\boldsymbol{z} \in t^{-1} V \cap D$ we have

$$(1-t)\,\boldsymbol{o} + t\boldsymbol{z} \in V \cap D,$$

whence

$$b \leqq f((1-t)\,\boldsymbol{o} + t\boldsymbol{z}) \leqq (1-t)\, f(\boldsymbol{o}) + t f(\boldsymbol{z}),$$

that is

$$f(\boldsymbol{z}) \geqq t^{-1} b.$$

If M is a bounded set, then $M \subseteq t^{-1} V$ for some t, where $0 < t \leqq 1$. In particular, for every $\boldsymbol{y} \in \bar{D}$ there exists a t, where $0 < t \leqq 1$. such that $\boldsymbol{y} \in t^{-1} V$. And $t^{-1} V$ is a neighbourhood of $\boldsymbol{y}$. Hence, the assertions (i) and (ii) follow from what has been proved above.

Obviously 2.6 (ii) is equivalent to

2.6. (iii) *If (D, f) is a convex function, then either*

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) > -\infty \qquad \textit{for all } \boldsymbol{x} \in \bar{D}$$

or

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) = -\infty \qquad \textit{for all } \boldsymbol{x} \in \bar{D}.$$

Remark to 2.6 (iii). In finite-dimensional spaces we always have the first alternative, whereas in infinite-dimensional spaces the second one may occur. Every non-continuous linear function ξ in a topological vector space $E[\mathfrak{T}]$ provides an example, since for all $c \in R$ the set $\{\boldsymbol{x} \mid \xi \boldsymbol{x} = c\}$ is dense in $E[\mathfrak{T}]$.

Next we prove (cf. [2] p. 92).

2.7. *Let (D, f) be a convex function, and let D be open. If there exists a point $\boldsymbol{x} \in D$ such that f is bounded above in some neighbourhood of $\boldsymbol{x}$, then f is continuous on D.*

Proof. Obviously we may assume that $\boldsymbol{x} = \boldsymbol{o}$ and $f(\boldsymbol{o}) = 0$. Let $a \in R$ and $V \in \mathfrak{V}$, V symmetric, be such that $f(\boldsymbol{z}) \leqq a$ for all $\boldsymbol{z} \in V$. Consider an ε such that $0 < \varepsilon < 1$. If $\boldsymbol{z} \in \varepsilon V$, then

$$f(\boldsymbol{z}) = f\left((1-\varepsilon)\,\boldsymbol{o} + \varepsilon\left(\frac{1}{\varepsilon}\boldsymbol{z}\right)\right) \leqq (1-\varepsilon)\,f(\boldsymbol{o}) + \varepsilon\, f\left(\frac{1}{\varepsilon}\boldsymbol{z}\right) \leqq \varepsilon\, a,$$

since $\varepsilon^{-1}\boldsymbol{z} \in V$. Further

$$0 = f(\boldsymbol{o}) = f\left(\frac{1}{1+\varepsilon}\boldsymbol{z} + \left(1 - \frac{1}{1+\varepsilon}\right)\left(-\frac{1}{\varepsilon}\boldsymbol{z}\right)\right) \leqq \frac{1}{1+\varepsilon} f(\boldsymbol{z}) + \left(1 - \frac{1}{1+\varepsilon}\right) f\left(-\frac{1}{\varepsilon}\boldsymbol{z}\right),$$

that is,

$$f(\boldsymbol{z}) \geqq -\varepsilon\, f\left(-\frac{1}{\varepsilon}\boldsymbol{z}\right) \geqq -\varepsilon\, a,$$

since $-\varepsilon^{-1}\boldsymbol{z} \in V$. Hence we have proved that $\boldsymbol{z} \in \varepsilon V$ implies $|f(\boldsymbol{z})| \leqq \varepsilon a$. This shows that f is continuous at $\boldsymbol{x}$.

Let $\boldsymbol{y}$ be in D. We shall prove that $\boldsymbol{y}$ has a neighbourhood in which f is bounded above. This will complete the proof of 2.7. Since D is open, there exists a $\varrho > 1$ such that $\varrho\boldsymbol{y} \in D$. Let $\boldsymbol{z}$ be in $\boldsymbol{y} + (1-\varrho^{-1})\,V$, where V has the same meaning as above. Then

$$\boldsymbol{z} = \boldsymbol{y} + \left(1 - \frac{1}{\varrho}\right)\boldsymbol{y}_0 = \frac{1}{\varrho}(\varrho\boldsymbol{y}) + \left(1 - \frac{1}{\varrho}\right)\boldsymbol{y}_0$$

with $\boldsymbol{y}_0 \in V \subseteqq D$. Since $0 < \varrho^{-1} < 1$, this implies $\boldsymbol{z} \in D$. Hence $\boldsymbol{y} + (1-\varrho^{-1})\,V \subseteqq D$. Further

$$f(\boldsymbol{z}) \leqq \frac{1}{\varrho} f(\varrho\boldsymbol{y}) + \left(1 - \frac{1}{\varrho}\right) f(\boldsymbol{y}_0) \leqq \frac{1}{\varrho} f(\varrho\boldsymbol{y}) + \left(1 - \frac{1}{\varrho}\right) a,$$

which proves that f is bounded above in $\boldsymbol{y} + (1-\varrho^{-1})V$.

2.8. *If (D, f) is a convex function, and D has a non-empty relative interior $\mathring{D}$, then the following statements are equivalent*:

a) *At least one $\boldsymbol{x} \in \mathring{D}$ has a neighbourhood in D in which f is bounded above.*

b) *Every $\boldsymbol{x} \in \mathring{D}$ has a neighbourhood in D in which f is bounded above.*

c) *f is continuous at at least one $\boldsymbol{x} \in \mathring{D}$.*

d) *f is continuous on $\mathring{D}$.*

e) *$[D, f]$ has a non-empty relative interior.*

Proof. We may assume that $m(D)$ is a subspace. Since $\mathring{D}$ is open and convex in $m(D)$, we may apply 2.7 to the function $(\mathring{D}, f)$, i.e. the restriction of f to $\mathring{D}$. This yields the equivalence of the statements a), b), c) and d). Since $m([D, f]) = m(D) \times R$, the statements a) and e) are obviously equivalent. Thus 2.8 is proved.

2.9. *If (D, f) is a convex function, and $[D, f]$ has a non-empty relative interior $[D, \overset{\circ}{f}]$, then D has a non-empty relative interior $\overset{\circ}{D}$, and the projection mapping $E \times R \rightarrow E$ maps $[D, \overset{\circ}{f}]$ onto $\overset{\circ}{D}$.*

Proof. This is an immediate consequence of 2.8.

Finally, we shall prove a result concerning convex functions in Banach spaces.

2.10. *Let (D, f) be a lower semi-continuous convex function in a Banach space, and let D have a non-empty relative interior $\overset{\circ}{D}$. Then f is continuous on $\overset{\circ}{D}$.*

Proof. We shall use the following lemma (cf. [7] p. 45):

Let S be a complete metric space and φ a lower semi-continuous function on S. Further, let T denote the set of points in S having a neighbourhood in which φ is bounded above. Then T is dense in S.

There exists a closed set $S \subseteq \overset{\circ}{D}$ with a non-empty interior $\overset{\circ}{S}$ in $m(D)$. Applying the lemma to the restriction of f to S, we obtain that $\overset{\circ}{S}$ contains a point $\boldsymbol{x}$ such that f is bounded above in a neighbourhood $(\boldsymbol{x}+V) \cap S$ of $\boldsymbol{x}$ in S. But $(\boldsymbol{x}+V) \cap S$ is also a neighbourhood of $\boldsymbol{x}$ in D. Hence the assertion follows from 2.8.

3. Conjugate functions

Let E_1 and E_2 be two vector spaces in duality under $\mathfrak{B}$, and let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be admissible topologies on E_1 and E_2. Then $E_2 = (E_1[\mathfrak{T}_1])'$ and $E_1 = (E_2[\mathfrak{T}_2])'$. Further $E_1 \times R$ and $E_2 \times R$ are in duality under $\tilde{\mathfrak{B}}((\xi, \xi), (\boldsymbol{x}, x)) = \mathfrak{B}(\xi, \boldsymbol{x}) + \xi x = \xi \boldsymbol{x} + \xi x$, and the product space topologies are admissible. So $(E_1[\mathfrak{T}_1] \times R)' = E_2 \times R$ and $(E_2[\mathfrak{T}_2] \times R)' = E_1 \times R$.

For a function (D, f) in E_1 we define

$$D' = \left\{ \xi \in E_2 \,\middle|\, \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) < \infty \right\},$$

$$f'(\xi) = \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) \qquad \text{for } \xi \in D'.$$

The set D' may be empty. If D' is not empty, the function $(D, f)' = (D', f')$ in E_2 is called *the conjugate of* (D, f).

If (Γ, φ) is a function in E_2, the conjugate (Γ', φ') in E_1 is defined analogously. In particular we have a *second conjugate* $(D, f)''$ of a function (D, f) in E_1, namely the conjugate of (D', f'). Setting $(D')' = D''$, $(f')' = f''$, we have for the second conjugate $(D, f)'' = ((D')', (f')') = (D'', f'')$

$$D'' = \left\{ \boldsymbol{x} \in E_1 \,\middle|\, \sup_{\xi \in D'} (\xi \boldsymbol{x} - f'(\xi)) < \infty \right\},$$

$$f''(\boldsymbol{x}) = \sup_{\xi \in D'} (\xi \boldsymbol{x} - f'(\xi)) \qquad \text{for } \boldsymbol{x} \in D''.$$

Hereafter the meaning of $(D^{(n)}, f^{(n)})$, $n \in Z_+$, is clear.

In the set of functions in a vector space a partial order relation is defined by

$$(D_1, f_1) \leq (D_2, f_2) \text{ if and only if } [D_2, f_2] \subseteq [D_1, f_1].$$

Then $(D_1, f_1) \leq (D_2, f_2)$ holds if and only if $D_2 \subseteq D_1$ and $f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in D_2$. Accordingly, (D_1, f_1) is called a *minorant* of (D_2, f_2), and (D_2, f_2) a *majorant* of (D_1, f_1), when $(D_1, f_1) \leq (D_2, f_2)$.

3.1. *If (D, f) has a conjugate, and $(D, f) \leq (D_1, f_1)$, then (D_1, f_1) has a conjugate, and $(D_1', f_1') \leq (D', f')$.*

Proof. Let ξ be in D'. Then

$$f'(\xi) = \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) \geq \sup_{\boldsymbol{x} \in D_1} (\xi \boldsymbol{x} - f_1(\boldsymbol{x})),$$

which implies $\xi \in D_1'$ and $f_1'(\xi) \leq f'(\xi)$. This proves 3.1.

3.2. *If (D, f) has a conjugate, then it has a conjugate of any order $n \in Z_+$. Further $(D'', f'') \leq (D, f)$, and $(D^{(n)}, f^{(n)})$ equals (D', f') for n odd, (D'', f'') for n even.*

Proof. Let $\boldsymbol{x}$ be in D. Then we have $f(\boldsymbol{x}) \geq \xi \boldsymbol{x} - f'(\xi)$ for all $\xi \in D'$, which implies $\boldsymbol{x} \in D''$ and $f(\boldsymbol{x}) \geq f''(\boldsymbol{x})$. Hence, (D, f) has a second conjugate, and $(D'', f'') \leq (D, f)$. Since (D'', f'') is the conjugate of (D', f'), the same argument applied to (D', f') shows that (D, f) has a third conjugate, and $(D''', f''') \leq (D', f')$. In particular, (D'', f'') has a conjugate, and since $(D'', f'') \leq (D, f)$, 3.1 yields $(D', f') \leq (D''', f''')$. Hence $(D''', f''') = (D', f')$. Hereafter, the unproved part of 3.2 follows by induction.

A closed hyperplane in $E_1[\mathfrak{T}_1] \times R$ is a set of the form

$$\{(\boldsymbol{x}, x) \in E_1 \times R \mid \eta \boldsymbol{x} + \eta x = c\},$$

where $(\eta, \eta) \in (E_1[\mathfrak{T}_1] \times R)' = E_2 \times R$ and $c \in R$, that is the set of those points $(\boldsymbol{x}, x)$ at which a continuous linear function (η, η) takes the value c. For the sake of convenience we specify the hyperplane by its equation $\eta \boldsymbol{x} + \eta x = c$. A hyperplane $\eta \boldsymbol{x} + \eta x = c$ in $E_1[\mathfrak{T}_1] \times R$ is called vertical if $\eta = 0$, non-vertical if $\eta \neq 0$.

Let M be a non-empty subset of $E_1[\mathfrak{T}_1] \times R$. By a *barrier* of M we mean a non-vertical closed hyperplane such that M is contained in one of the two closed halfspaces determined by the hyperplane. This means that if $\eta \boldsymbol{x} + \eta x = c$ is a barrier of M, we have either $\eta \boldsymbol{x} + \eta x \geq c$ or $\eta \boldsymbol{x} + \eta x \leq c$ for all

$(\boldsymbol{x}, x) \in M$. If M is of the form $[D, f]$, the equation of a barrier $\boldsymbol{\eta}\boldsymbol{x} + \eta x = c$ can always be written in the form $\boldsymbol{\xi}\boldsymbol{x} - x = \xi$, $(\boldsymbol{\xi}, -1) \in E_2 \times R$, $\xi \in R$, and such that $\boldsymbol{\xi}\boldsymbol{x} - x \leq \xi$ for all $(\boldsymbol{x}, x) \in M = [D, f]$. For, division by $-\eta$ gives an equation of the form $\boldsymbol{\xi}\boldsymbol{x} - x = \xi$, and $\boldsymbol{\xi}\boldsymbol{x} - x \geqq \xi$ for all $(\boldsymbol{x}, x) \in [D, f]$ is impossible, since $(\boldsymbol{x}, x) \in [D, f]$ and $k \in R_+$ implies $(\boldsymbol{x}, x + k) \in [D, f]$.

A barrier of the set $[D, f]$ will also be referred to as a barrier of (D, f).

The previously defined barriers of a function (D, f) have a close relation to the conjugate of (D, f). Let (D, f) be a function in E_1, and let $\boldsymbol{\xi}\boldsymbol{x} - x = \xi$, where $(\boldsymbol{\xi}, -1) \in E_2 \times R$, $\xi \in R$, be a barrier of (D, f). Then $\xi \geqq \boldsymbol{\xi}\boldsymbol{x} - x$ for all $(\boldsymbol{x}, x) \in [D, f]$. In particular, $\xi \geqq \boldsymbol{\xi}\boldsymbol{x} - f(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$. Hence $\boldsymbol{\xi} \in D'$ and $\xi \geqq f'(\boldsymbol{\xi})$, and so $(\boldsymbol{\xi}, \xi) \in [D', f']$. Conversely, let (D, f) have a conjugate, and let $(\boldsymbol{\xi}, \xi)$ be in $[D', f']$. Then we have

$$\xi \geqq f'(\boldsymbol{\xi}) \geqq \boldsymbol{\xi}\boldsymbol{x} - f(\boldsymbol{x}) \geqq \boldsymbol{\xi}\boldsymbol{x} - x$$

for all $(\boldsymbol{x}, x) \in [D, f]$, which shows that $\boldsymbol{\xi}\boldsymbol{x} - x = \xi$ is a barrier of (D, f). Hence, we have proved

3.3. *A function (D, f) has a conjugate if and only if it has a barrier. If (D, f) has a conjugate, then the point $(\boldsymbol{\xi}, \xi)$ is in $[D', f']$ if and only if the hyperplane $\{(\boldsymbol{x}, x) \mid \boldsymbol{\xi}\boldsymbol{x} - x = \xi\}$ is a barrier of (D, f).*

Because of the duality, 3.3 is also valid for functions in E_2, in particular for (D', f'). This gives

3.4. *If (D, f) has a conjugate, then the point $(\boldsymbol{x}, x)$ is in $[D'', f'']$ if and only if the hyperplane $\{(\boldsymbol{\xi}, \xi) \mid \boldsymbol{\xi}\boldsymbol{x} - \xi = x\}$ is a barrier of (D', f').*

From 3.3 and 3.4 follows immediately

3.5. *If (D, f) has a conjugate, then*

$$\text{(i)} \quad [D', f'] = \bigcap_{(\boldsymbol{x}, x) \in [D, f]} \{(\boldsymbol{\xi}, \xi) \mid \boldsymbol{\xi}\boldsymbol{x} - \xi \leqq x\}.$$

$$\text{(ii)} \quad [D'', f''] = \bigcap_{(\boldsymbol{\xi}, \xi) \in [D', f']} \{(\boldsymbol{x}, x) \mid \boldsymbol{\xi}\boldsymbol{x} - x \leqq \xi\}.$$

Consequently the sets $[D', f']$ and $[D'', f'']$ are both intersections of closed halfspaces, and thus convex and closed. Thus, by 2.1 and 2.2, we have

3.6. Theorem. *If (D, f) has a conjugate, then (D', f') and (D'', f'') are closed convex functions.*

Concerning the existence of conjugates we have

3.7. Theorem. *If (D, f) is a closed convex function, then it has a conjugate.*

Proof. For $\boldsymbol{y} \in D$ and $k \in R_+$ the point $(\boldsymbol{y}, f(\boldsymbol{y}) - k)$ is not in $[D, f]$. Consequently (cf. theorem 1.1) there exists a closed hyperplane $\boldsymbol{\eta x} + \eta x = c$, $(\boldsymbol{\eta}, \eta) \in E_2 \times R$, $c \in R$, that strictly separates $[D, f]$ and the point $(\boldsymbol{y}, f(\boldsymbol{y}) - k)$. We can assume $\boldsymbol{\eta x} + \eta x < c$ for all $(\boldsymbol{x}, x) \in [D, f]$ and $\boldsymbol{\eta y} + \eta (f(\boldsymbol{y}) - k) > c$. Since $\boldsymbol{y} \in D$, it follows that $\eta \neq 0$. Hence $\boldsymbol{\eta x} + \eta x = c$ is a barrier of (D, f).

In 3.2 we showed that $(D'', f'') \leq (D, f)$. A fundamental question is under which conditions $(D'', f'') = (D, f)$ holds. By 3.6, it is necessary that (D, f) be convex and closed. Theorem 3.10 below states that this is also sufficient. In the proof we shall use

3.8. *Let M be a closed convex subset of $E_1 \times R$. If M has at least one barrier, then M is the intersection of all closed halfspaces containing M and bounded by barriers of M.*

Proof. Being convex and closed, M is the intersection of all closed halfspaces containing it (cf. theorem 1.2). Thus we have to show that if there exists a vertical closed hyperplane separating M and the point $(\boldsymbol{y}, y)$, and not containing $(\boldsymbol{y}, y)$, then there exists a non-vertical closed hyperplane with the same property. Let $\boldsymbol{\zeta x} = c$, $(\boldsymbol{\zeta}, 0) \in E_2 \times R$, $c \in R$, be a vertical closed hyperplane in $E_1 \times R$ such that $\boldsymbol{\zeta y} > c$ and $\boldsymbol{\zeta x} \leq c$ for all $\boldsymbol{x} \in p(M)$, p denoting the projection mapping $E_1 \times R \to E_1$. Further let $\boldsymbol{\eta x} + \eta x = \gamma$, $(\boldsymbol{\eta}, \eta) \in E_2 \times R$, $\eta \neq 0$, $\gamma \in R$, be a barrier of M. We may assume $\boldsymbol{\eta x} + \eta x \leq \gamma$ for all $(\boldsymbol{x}, x) \in M$. Now, for every $t \in R_+$

$$\{(\boldsymbol{x}, x) \in E_1 \times R \mid (\boldsymbol{\eta} + t\boldsymbol{\zeta})\, \boldsymbol{x} + \eta x = \gamma + tc\}$$

is a closed non-vertical hyperplane such that

$$(\boldsymbol{\eta} + t\boldsymbol{\zeta})\, \boldsymbol{x} + \eta x \leq \gamma + tc$$

for all $(\boldsymbol{x}, x) \in M$. If

$$(\boldsymbol{\eta} + t\boldsymbol{\zeta})\, \boldsymbol{y} + \eta y \leq \gamma + tc,$$

then

$$t\,(\boldsymbol{\zeta y} - c) \leq \gamma - \boldsymbol{\eta y} - \eta y.$$

But this cannot be true for all $t \in R_+$ since $\boldsymbol{\zeta y} > c$. Consequently, there exists a t_0 such that

$$(\boldsymbol{\eta} + t_0 \boldsymbol{\zeta})\, \boldsymbol{y} + \eta y > \gamma + t_0 c,$$

and so the non-vertical hyperplane

$$\{(\boldsymbol{x}, x) \in E_1 \times R \mid (\boldsymbol{\eta} + t_0 \boldsymbol{\zeta})\, \boldsymbol{x} + \eta x = \gamma + t_0 c\}$$

separates M and the point $(\boldsymbol{y}, y)$, and does not contain $(\boldsymbol{y}, y)$.

3.9. Theorem. *If (D, f) has a conjugate, then*

$$[D'', f''] = \overline{\operatorname{conv}\,[D, f]}.$$

Proof. By 3.3 and 3.5 (ii), $[D'', f'']$ is the intersection of all closed halfspaces containing $[D, f]$ and bounded by barriers of $[D, f]$. Since the barriers of $[D, f]$ are identical with the barriers of $\overline{\operatorname{conv}\,[D, f]}$, it follows from 3.8 that $\overline{\operatorname{conv}\,[D, f]}$ is the intersection of the same closed halfspaces as $[D'', f'']$, which proves the statement.

Since for a closed convex function (D, f)

$$[D, f] = \overline{\operatorname{conv}\,[D, f]},$$

3.9 yields the theorem previously mentioned:

3.10. Theorem. *If (D, f) is convex and closed, then $(D, f) = (D'', f'')$.*

We note that, under the assumptions of 3.10, the existence of (D'', f'') is ensured by 3.7.

As easily seen, there exists a closed convex minorant of the function (D, f) if and only if (D, f) has a conjugate. In that case there even exists a greatest closed convex minorant of (D, f) namely (D'', f''). For, let (D_1, f_1) be a closed convex minorant. Then $[D_1, f_1]$ is a closed convex set, and $[D, f] \subseteq [D_1, f_1]$. By 3.9, this implies $[D'', f''] \subseteq [D_1, f_1]$. So we have proved

3.11. Theorem. *If (D, f) has a conjugate, then (D'', f'') is its greatest closed convex minorant.*

The question arises which functions do have conjugates. A necessary condition is that

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) > -\infty \text{ for all } \boldsymbol{x} \in \overline{D}.$$

For, let $\xi \boldsymbol{x} - x = \xi$ be a barrier. Since ξ is continuous, it is possible for every $\boldsymbol{x} \in E$ to find a $V \in \mathfrak{B}$ such that $\xi \boldsymbol{z} \geqq \xi \boldsymbol{x} - 1$ for all $\boldsymbol{z} \in \boldsymbol{x} + V$. From this we deduce

$$\begin{aligned}\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) &\geqq \inf\{f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (D \setminus \{\boldsymbol{x}\})\} \\ &\geqq \inf\{\xi \boldsymbol{z} - \xi \mid \boldsymbol{z} \in \boldsymbol{x} + V\} \geqq \xi \boldsymbol{x} - 1 - \xi > -\infty.\end{aligned}$$

We shall prove that, for convex functions, this is also sufficient.

3.12. Theorem. (i) *Let (D, f) be a convex function. Then (D, f) has a conjugate if (and only if)*

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) > -\infty \qquad \textit{for all } \boldsymbol{x} \in \overline{D}.$$

In view of 2.6. (iii), we may also formulate the theorem in the following way:

3.12. Theorem. (ii) *Let (D, f) be a convex function, and let $\boldsymbol{x}$ be an arbitrary point in $\bar{D}$. Then (D, f) has a conjugate if (and only if)*

$$\liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) > -\infty .$$

Proof. Let (D, f) be a convex function such that for all $\boldsymbol{x} \in \bar{D}$

$$\liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) > -\infty .$$

Then the function

$$\hat{f}(\boldsymbol{x}) = \liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z})$$

is well-defined on the set

$$\hat{D} = \left\{ \boldsymbol{x} \in \bar{D} \mid \liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) < \infty \right\}.$$

a) *$(\hat{D}, \hat{f})$ is a convex function.*

Proof of a). Let $\boldsymbol{y}_0$ and $\boldsymbol{y}_1$ be in $\hat{D}$, and consider $\boldsymbol{y}_t = (1-t)\,\boldsymbol{y}_0 + t\boldsymbol{y}_1$, where $0 \leq t \leq 1$. Let $\boldsymbol{y}_t + V$ be a convex neighbourhood of $\boldsymbol{y}_t$. From the definition of $(\hat{D}, \hat{f})$ it follows that for every $\varepsilon \in R_+$ there exist a $\boldsymbol{z}_0 \in (\boldsymbol{y}_0 + V) \cap (D \setminus \{\boldsymbol{y}_0\})$ and a $\boldsymbol{z}_1 \in (\boldsymbol{y}_1 + V) \cap (D \setminus \{\boldsymbol{y}_1\})$ such that $f(\boldsymbol{z}_0) \leqq \hat{f}(\boldsymbol{y}_0) + \varepsilon$ and $f(\boldsymbol{z}_1) \leqq \hat{f}(\boldsymbol{y}_1) + \varepsilon$.

Since $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$ are in D, the point $\boldsymbol{z}_t = (1-t)\,\boldsymbol{z}_0 + t\boldsymbol{z}_1$ is in D, and

$$f(\boldsymbol{z}_t) \leqq (1-t)\,f(\boldsymbol{z}_0) + t f(\boldsymbol{z}_1) \leqq (1-t)\,\hat{f}(\boldsymbol{y}_0) + t\hat{f}(\boldsymbol{y}_1) + \varepsilon .$$

Further $\boldsymbol{z}_t \in (1-t)(\boldsymbol{y}_0 + V) + t(\boldsymbol{y}_1 + V) = \boldsymbol{y}_t + V$. Hence

$$\inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{y}_t + V) \cap D \right\} \leqq (1-t)\,\hat{f}(\boldsymbol{y}_0) + t\hat{f}(\boldsymbol{y}_1).$$

This implies

$$\inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{y}_t + V) \cap (D \setminus \{\boldsymbol{y}_t\}) \right\} \leqq (1-t)\,\hat{f}(\boldsymbol{y}_0) + t\hat{f}(\boldsymbol{y}_1),$$

since, if $\boldsymbol{y}_t \in D$,

$$f(\boldsymbol{y}_t) \geqq \liminf_{\boldsymbol{z}\to\boldsymbol{y}_t} f(\boldsymbol{z}) \geqq \inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{y}_t + V) \cap (D \setminus \{\boldsymbol{y}_t\}) \right\}.$$

Consequently

$$\liminf_{\boldsymbol{z}\to\boldsymbol{y}_t} f(\boldsymbol{z}) \leqq (1-t)\,\hat{f}(\boldsymbol{y}_0) + t\hat{f}(\boldsymbol{y}_1),$$

which shows that $(\hat{D}, \hat{f})$ is convex.

b) *$(\hat{D}, \hat{f})$ is a minorant of (D, f).*

Proof of b). From the convexity of (D, f) it follows that $\liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z})$

$\leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$. This implies $D \subseteq \hat{D}$ and $f(\boldsymbol{x}) \geq \hat{f}(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$. Hence $(\hat{D}, \hat{f})$ is a minorant of (D, f).

c) *For all* $\boldsymbol{x} \in \overline{D}$ *we have* $\liminf_{\boldsymbol{z} \to \boldsymbol{x}} \hat{f}(\boldsymbol{z}) = \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z})$.

Proof of c). Let $\boldsymbol{x}$ be in $\overline{D}$. Since $(\hat{D}, \hat{f})$ is a minorant of (D, f), we have

$$\inf \left\{ \hat{f}(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (\hat{D} \setminus \{\boldsymbol{x}\}) \right\}$$
$$\leq \inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (D \setminus \{\boldsymbol{x}\}) \right\}$$

for all $V \in \mathfrak{V}$. This implies

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} \hat{f}(\boldsymbol{z}) \leq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}).$$

To prove the reversed inequality we consider an arbitrary $\boldsymbol{z}_0 \in (\boldsymbol{x} + V) \cap (\hat{D} \setminus \{\boldsymbol{x}\})$, where $V \in \mathfrak{V}$ is assumed to be open. As the topology $\mathfrak{T}_1$ on E_1 is Hausdorff, and V is open, $(\boldsymbol{x} + V) \setminus \{\boldsymbol{x}\}$ is a neighbourhood of $\boldsymbol{z}_0$. Let $\varepsilon \in R_+$ be given. Since $\hat{f}(\boldsymbol{z}_0) = \liminf_{\boldsymbol{z} \to \boldsymbol{z}_0} f(\boldsymbol{z})$, there exists a $\boldsymbol{z}_1$ such that

$$\boldsymbol{z}_1 \in ((\boldsymbol{x} + V) \setminus \{\boldsymbol{x}\}) \cap (D \setminus \{\boldsymbol{z}_0\}) \subseteq (\boldsymbol{x} + V) \cap (D \setminus \{\boldsymbol{x}\})$$

and

$$f(\boldsymbol{z}_1) \leq \hat{f}(\boldsymbol{z}_0) + \varepsilon.$$

This proves

$$\inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (D \setminus \{\boldsymbol{x}\}) \right\} \leq \hat{f}(\boldsymbol{z}_0).$$

Hence

$$\inf \left\{ f(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (D \setminus \{\boldsymbol{x}\}) \right\}$$
$$\leq \inf \left\{ \hat{f}(\boldsymbol{z}) \mid \boldsymbol{z} \in (\boldsymbol{x} + V) \cap (\hat{D} \setminus \{\boldsymbol{x}\}) \right\},$$

and thus

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) \leq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} \hat{f}(\boldsymbol{z}).$$

d) $(\hat{D}, \hat{f})$ *is a closed function.*

Proof of d). Let $\boldsymbol{x}$ be a point in $\overline{\hat{D}} = \overline{D}$ such that $\liminf_{\boldsymbol{z} \to \boldsymbol{x}} \hat{f}(\boldsymbol{z}) < \infty$. From c) and the definition of $(\hat{D}, \hat{f})$ it follows that $\boldsymbol{x} \in \hat{D}$ and $\hat{f}(\boldsymbol{x}) = \liminf_{\boldsymbol{z} \to \boldsymbol{x}} \hat{f}(\boldsymbol{z})$. Hence, $(\hat{D}, \hat{f})$ is closed.

Now, from a), b) and d) it follows that (D, f) has a closed convex minorant. Thus, (D, f) has a conjugate.

3.13. Theorem. *If (D, f) is a convex function, and it has a conjugate, then the second conjugate (D'', f'') is determined by*

$$D'' = \left\{ \boldsymbol{x} \in \overline{D} \mid \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) < \infty \right\},$$

$$f''(\boldsymbol{x}) = \liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) \qquad \text{for } \boldsymbol{x} \in D''.$$

Consequently, if f is lower semi-continuous at $\boldsymbol{x} \in D$, then $f''(\boldsymbol{x}) = f(\boldsymbol{x})$.

Proof. Let (D, f) be convex. If it has a conjugate, then $\liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) > -\infty$ for all $\boldsymbol{x} \in \bar{D}$. Hence, we may define $(\hat{D}, \hat{f})$ as in the preceding proof and prove that $(\hat{D}, \hat{f})$ is a closed convex minorant of (D, f). In fact $(\hat{D}, \hat{f})$ is the greatest closed convex minorant of (D, f). For, let (D_1, f_1) be a closed minorant of (D, f). Then for every $\boldsymbol{x} \in \hat{D}$

$$\liminf_{\boldsymbol{z}\to\boldsymbol{x}} f_1(\boldsymbol{z}) \leq \liminf_{\boldsymbol{z}\to\boldsymbol{x}} f(\boldsymbol{z}) = \hat{f}(\boldsymbol{x}) < \infty .$$

Since (D_1, f_1) is closed, this implies $\boldsymbol{x} \in D_1$ and $f_1(\boldsymbol{x}) \leq \hat{f}(\boldsymbol{x})$, i.e. (D_1, f_1) is a minorant of $(\hat{D}, \hat{f})$. Thus, by 3.11, we have $(\hat{D}, \hat{f}) = (D'', f'')$, which proves the theorem.

4. The conjugates of functions derived from others

In this section we shall consider questions of the following kind. Suppose, a function (D, f) is derived in a certain way from functions (D_i, f_i), $i \in J$, where J is an index set. Is the conjugate (D', f') determined by the conjugates (D_i', f_i'), and in the affirmative case, in what manner?

4.1. Theorem. *Let the function (D_0, f_0) have a conjugate (D_0', f_0'), and let (D, f) be defined by*

$$D = \boldsymbol{x}_0 + lD_0$$

$$f(\boldsymbol{x}) = kf_0(l^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)) + \xi_0 \boldsymbol{x} + h,$$

where $\boldsymbol{x}_0 \in E_1$, $\xi_0 \in E_2$, and h, k and l are reals such that $k > 0$ and $l \neq 0$. Then (D, f) has a conjugate (D', f') which is determined by

$$D' = \xi_0 + kl^{-1} D_0'$$

$$f'(\xi) = kf_0'(lk^{-1}(\xi - \xi_0)) + (\xi - \xi_0)\boldsymbol{x}_0 - h.$$

Proof. For all $\xi \in E_2$ we have

$$\sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) = \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - kf_0(l^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)) - \xi_0 \boldsymbol{x} - h)$$
$$= k \cdot \sup_{\boldsymbol{y} \in D_0} (lk^{-1}(\xi - \xi_0)\boldsymbol{y} - f_0(\boldsymbol{y})) + (\xi - \xi_0)\boldsymbol{x}_0 - h.$$

This proves the theorem.

4.2. Theorem. *Let* (D_1, f_1) *and* (D_2, f_2) *be closed convex functions such that* $D_1 \cap D_2 \neq \emptyset$, *and let* (D, f) *be defined by*

$$D = D_1 \cap D_2, \quad f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}).$$

Then we have:

(i) (D, f) *is a closed convex function.*

(ii) $[D', f'] = \overline{[D_1', f_1'] + [D_2', f_2']}$.

(iii) $D_1' + D_2' \subseteq D' \subseteq \overline{D_1' + D_2'}$.

(iv) *If* $[D', f']$ *has a non-empty relative interior, then*

$$f'(\boldsymbol{\xi}) = \inf \{ f_1'(\boldsymbol{\xi}_1) + f_2'(\boldsymbol{\xi}_2) \mid \boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \boldsymbol{\xi}_1 \in D_1', \boldsymbol{\xi}_2 \in D_2' \}$$

for all $\boldsymbol{\xi}$ *in the relative interior of* D'.

Proof. (i) The convexity is obvious. Since $\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_1(\boldsymbol{z}) > -\infty$ for all $\boldsymbol{x} \in \bar{D}_1$, and $\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_2(\boldsymbol{z}) > -\infty$ for all $\boldsymbol{x} \in \bar{D}_2$, the expression

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_1(\boldsymbol{z}) + \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_2(\boldsymbol{z})$$

is well-defined for all $\boldsymbol{x} \in \bar{D} \subseteq \bar{D}_1 \cap \bar{D}_2$. Further, for all $\boldsymbol{x} \in \bar{D}$

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_1(\boldsymbol{z}) + \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_2(\boldsymbol{z}) \leq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}).$$

Let $\boldsymbol{x} \in \bar{D}$ be such that $\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}) < \infty$. Then, by the preceding inequality,

$$\liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_1(\boldsymbol{z}) + \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f_2(\boldsymbol{z}) < \infty.$$

As (D_1, f_1) and (D_2, f_2) are closed, this implies $\boldsymbol{x} \in D_1 \cap D_2 = D$, and

$$f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}) \leq \liminf_{\boldsymbol{z} \to \boldsymbol{x}} f(\boldsymbol{z}).$$

Thus (D, f) is closed.

(ii) $D_1 \cap D_2 \neq \emptyset$ implies $[D_1, f_1] \cap [D_2, f_2] \neq \emptyset$. Hence $[D_1', f_1']$ and $[D_2', f_2']$ have a common barrier $\boldsymbol{\xi x} - \xi = x$. Then $\boldsymbol{\xi x} - \xi = 2x$ is a barrier of $[D_1', f_1'] + [D_2', f_2']$. This implies that the closed convex set $\overline{[D_1', f_1'] + [D_2', f_2']}$ is of the form $[\Gamma, \varphi]$, where (Γ, φ) is a closed convex function in E_2. Now $(\Gamma', \varphi') = (D, f)$, which may be proved in the following way. If $(\boldsymbol{x}, x) \in [\Gamma', \varphi']$, then

$$(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2)\boldsymbol{x} - (\xi_1 + \xi_2) \leq x$$

for all $(\boldsymbol{\xi}_1, \xi_1) \in [D_1', f_1']$, $(\boldsymbol{\xi}_2, \xi_2) \in [D_2', f_2']$, and this implies

$$\sup_{\xi \in D_1'} (\xi \boldsymbol{x} - f_1'(\xi)) + \sup_{\xi \in D_2'} (\xi \boldsymbol{x} - f_2'(\xi)) \leq x .$$

Hence $\boldsymbol{x} \in D_1'' \cap D_2'' = D_1 \cap D_2 = D$, and

$$x \geq f_1''(\boldsymbol{x}) + f_2''(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}) = f(\boldsymbol{x}),$$

that is $(\boldsymbol{x}, x) \in [D, f]$. Conversely, for every $(\boldsymbol{x}, x) \in [D, f]$ we have

$$(\xi_1 + \xi_2) \boldsymbol{x} - (\xi_1 + \xi_2) \leq f_1''(\boldsymbol{x}) + f_2''(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}) = f(\boldsymbol{x}) \leq x$$

for all $(\xi_1, \xi_1) \in [D_1', f_1']$, $(\xi_2, \xi_2) \in [D_2', f_2']$. Thus $\xi \boldsymbol{x} - \xi \leq x$ for all $(\xi, \xi) \in [D_1', f_1'] + [D_2', f_2']$, and consequently then also for all $(\xi, \xi) \in \overline{[D_1', f_1'] + [D_2', f_2']} = [\Gamma, \varphi]$. This shows that $(\boldsymbol{x}, x) \in [\Gamma', \varphi']$. Hence we have proved $(\Gamma', \varphi') = (D, f)$. As (Γ, φ) is convex and closed, this implies $(\Gamma, \varphi) = (D', f')$, which proves (ii).

(iii) This is an obvious consequence of (ii).

(iv) Since $\mathring{M} = \mathring{\overline{M}}$, M convex, $M \neq \emptyset$, it follows from (ii) that the relative interior $[D', {}^\circ f']$ of $[D', f']$ is equal to the relative interior of $[D_1', f_1'] + [D_2', f_2']$. (Likewise, we have by (iii) that the relative interior of D' is equal to the relative interior of $D_1' + D_2'$). Hence

$$[D', {}^\circ f'] \subseteq [D_1', f_1'] + [D_2', f_2'] \subseteq [D', f'].$$

Let ξ be in the relative interior of D'. Then, by 2.9, there exists a $\xi \in R$ such that $(\xi, \xi) \in [D', {}^\circ f']$. Since all points in $[D', f']$ can be reached from $[D', {}^\circ f']$, we have

$$(\xi, \zeta) \in [D', {}^\circ f'] \subseteq [D_1', f_1'] + [D_2', f_2']$$

for all $\zeta > f'(\xi)$. Thus

$$f'(\xi) = \inf \{ \zeta \mid (\xi, \zeta) \in [D_1', f_1'] + [D_2', f_2'] \},$$

which proves (iv).

In accordance with the partial order previously defined, a function (D, f) will be called a minorant of a set of functions $\{(D_i, f_i) \mid i \in J\}$ if $(D, f) \leq (D_i, f_i)$ for all $i \in J$. Analogously a majorant is defined.

If there exists a minorant of the set $\{(D_i, f_i) \mid i \in J\}$, then the function (D, f) defined by

$$D = \bigcup_{i \in J} D_i,$$

$$f(\boldsymbol{x}) = \inf \{ f_i(\boldsymbol{x}) \mid i \in J, \boldsymbol{x} \in D_i \}$$

is the greatest minorant. We shall denote this function by $\bigwedge_{i \in J}(D_i, f_i)$, or for the sake of brevity $\bigwedge(D_i, f_i)$, and the set $[D, f]$ by $\bigwedge_{i \in J}[D_i, f_i]$, or briefly $\bigwedge[D_i, f_i]$. (Similar abbreviations will be used below in connexion with the symbols $\hat{\bigwedge}$, $\bigvee$, $\bigcup$ and $\bigcap$). Clearly

$$\bigcup [D_i, f_i] \subseteq \bigwedge [D_i, f_i] \subseteq \overline{\bigcup [D_i, f_i]}.$$

If $\bigwedge(D_i, f_i)$ exists, then there exists a closed convex minorant of $\{(D_i, f_i)\}$ if and only if $\bigwedge(D_i, f_i)$ has a conjugate $(\bigwedge(D_i, f_i))'$. In that case there exists a greatest closed convex minorant, denoted $\hat{\bigwedge}(D_i, f_i)$, namely the second conjugate $(\bigwedge(D_i, f_i))''$ of $\bigwedge(D_i, f_i)$.

Of course, the greatest minorant of a set $\{(D_i, f_i)\}$ of convex functions need not be convex. However, if the set $\{(D_i, f_i)\}$ is totally ordered, $\bigwedge(D_i, f_i)$ is easily seen to be convex.

Suppose that there exists a majorant of $\{(D_i, f_i) \mid i \in J\}$. Then the function (D, f) defined by

$$D = \{\boldsymbol{x} \in \bigcap_{i \in J} D_i \mid \sup_{i \in J} f_i(\boldsymbol{x}) < \infty\},$$

$$f(\boldsymbol{x}) = \sup_{i \in J} f_i(\boldsymbol{x})$$

is the smallest majorant. This function (D, f) is denoted by $\bigvee(D_i, f_i)$ and the set $[D, f]$ by $\bigvee[D_i, f_i]$.

Obviously, $\bigvee(D_i, f_i)$ exists if and only if $\bigcap[D_i, f_i]$ is non-empty, and in that case

$$\bigvee[D_i, f_i] = \bigcap[D_i, f_i].$$

Hence, the smallest majorant of a set of convex or closed functions is convex or closed, respectively.

4.3. *Given a set of functions* $\{(D_i, f_i) \mid i \in J\}$.

(i) *If* $\bigwedge(D_i, f_i)$ *and* $(\bigwedge(D_i, f_i))'$ *exist, then* $(\bigwedge(D_i, f_i))' = \bigvee(D_i', f_i')$.

(ii) *If* $\bigvee(D_i, f_i)$ *and at least one* (D_i', f_i') *exist, then* $(\bigvee(D_i, f_i))' \leq \bigwedge(D_i', f_i')$ (where $\bigwedge(D_i', f_i')$ means the greatest minorant of those (D_i', f_i') which exist).

(iii) *If all* (D_i, f_i) *are convex and closed, and* $\bigvee(D_i, f_i)$ *exists, then* $(\bigvee(D_i, f_i))' = (\bigwedge(D_i', f_i'))''$.

In all cases the assumptions ensure the existence of the minorants, majorants and conjugates occurring in the statements.

Proof. (i) It is easily seen that the barriers of $\bigwedge(D_i, f_i)$ are precisely the common barriers of $\{(D_i, f_i)\}$. This implies that $\vee (D_i', f_i')$ exists and that (i) holds.

(ii) For all $j \in J$ we have $(D_j, f_j) \leq \vee (D_i, f_i)$. Hence the existence of at least one (D_j', f_j') implies that of $(\vee (D_i, f_i))'$, and we have $(\vee (D_i, f_i))' \leq (D_j', f_j')$ for every j for which (D_j', f_j') exists. This shows that $\wedge (D_i', f_i')$ exists and that (ii) holds.

(iii) Under the assumptions all (D_i', f_i') exist. From (ii) follows that $\wedge (D_i', f_i')$ and $(\vee (D_i, f_i))'$ exist and that $(\vee (D_i, f_i))' \leq \wedge (D_i', f_i')$. This implies that $(\wedge (D_i', f_i'))'$ exists, and (i) applied to $\{(D_i', f_i')\}$ then gives

$$(\wedge (D_i', f_i'))' = \vee (D_i'', f_i'') = \vee (D_i, f_i),$$

since all (D_i, f_i) are convex and closed. Hence $(\wedge (D_i', f_i'))'' = (\vee (D_i, f_j))'$.

4.4. Theorem. *Let $\{(D_i, f_i)\}$ be a set of closed convex functions. If $\wedge (D_i, f_i)$, $(\wedge (D_i, f_i))'$ and $\vee (D_i, f_i)$ exist, then*

$$(\hat{\wedge} (D_i, f_i))' = \vee (D_i', f_i'),$$

$$(\vee (D_i, f_i))' = \hat{\wedge} (D_i', f_i').$$

Proof. This is an obvious consequence of 4.3 (i) and (iii).

4.5. Theorem. *Let $\{(D_i, f_i)\}$ be a set of closed convex functions, and suppose that $\vee (D_i, f_i)$ exists. For $(D, f) = \vee (D_i, f_i)$ we then have*

(i) $[D', f'] = \overline{\operatorname{conv} (\cup [D_i', f_i'])}$.

(ii) $\operatorname{conv} (\cup D_i') \subseteq D' \subseteq \overline{\operatorname{conv} (\cup D_i')}$.

(iii) *If $[D', f']$ has a non-empty relative interior, then*

$$f'(\xi) = \inf \left\{ \sum_{\nu=1}^{n} \lambda_\nu f_{i_\nu}'(\xi_\nu) \,\middle|\, \xi = \sum_{\nu=1}^{n} \lambda_\nu \xi_\nu, \xi_\nu \in D_{i_\nu}', \lambda_\nu \geq 0, \sum_{\nu=1}^{n} \lambda_\nu = 1, n \in Z_+ \right\}$$

for all ξ in the relative interior of D'.

Proof. (i) All (D_i', f_i') and $\wedge (D_i', f_i')$ exist, and we have

$$\cup [D_i', f_i'] \subseteq \wedge [D_i', f_i'] \subseteq \overline{\cup [D_i', f_i']}.$$

This implies

$$\overline{\operatorname{conv} (\cup [D_i', f_i'])} = \overline{\operatorname{conv} (\wedge [D_i', f_i'])},$$

since $\overline{\operatorname{conv} M} = \overline{\operatorname{conv} \overline{M}}$ for any set M. Statement (i) then follows from 3.9 and 4.3 (iii).

(ii) This is a simple consequence of (i).

(iii) By (i) we have

$$[D',{}^{\circ}f'] \subseteq \operatorname{conv}(\cup [D'_i, f'_i]) \subseteq [D', f'],$$

$[D',{}^{\circ}f']$ denoting the relative interior of $[D', f']$. Let ξ be in the relative interior of D'. Then $(\xi, \zeta) \in [D',{}^{\circ}f']$ for all $\zeta > f'(\xi)$, that is

$$f'(\xi) = \inf\{\zeta \mid (\xi, \zeta) \in \operatorname{conv}(\cup [D'_i, f'_i])\}.$$

5. Convex functions with domains having non-empty relative interiors

As usual, E_1 and E_2 are vector spaces in duality. We first prove a result concerning the structure of closed convex functions.

5.1. *Let (D, f) be a closed convex function in E_1. There exists one and, obviously, only one subspace F_1 of E_1, called the linearity space of (D, f), with the following properties:*

(i) *F_1 is closed.*

(ii) *$D + F_1 = D$.*

(iii) *For every $\boldsymbol{x} \in D$*

$$f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}), \quad \boldsymbol{z} \in F_1,$$

is a continuous linear function on F_1, independent of $\boldsymbol{x}$.

(iv) *Every subspace of E_1 with the properties* (ii) *and* (iii) *is a subspace of F_1.*

(v) *If (Γ, φ) denotes the conjugate of (D, f) in E_2, then*

$$m(\Gamma) = \xi + F_1^{\perp}$$

for every $\xi \in \Gamma$.

Proof. We define $F_1 = (\Gamma - \xi_0)^{\perp}$, where $\xi_0 \in \Gamma$. Thus, a point $\boldsymbol{x} \in E_1$ is in F_1 if and only if it is a constant function on Γ. Obviously F_1 is a closed subspace. For $\boldsymbol{x} \in D$ and $\boldsymbol{z} \in F_1$ we have

$$\sup_{\xi \in \Gamma} (\xi(\boldsymbol{x}+\boldsymbol{z}) - \varphi(\xi)) = \sup_{\xi \in \Gamma} (\xi \boldsymbol{x} - \varphi(\xi)) + \xi_0 \boldsymbol{z}.$$

Hence $\boldsymbol{x}+\boldsymbol{z} \in D$ and $f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}) = \xi_0 \boldsymbol{z}$, which proves (ii) and (iii). Let $\boldsymbol{y}_0 \neq \boldsymbol{o}$ be in a subspace with the properties (ii) and (iii). Then there exists a continuous linear function η on the subspace generated by $\boldsymbol{y}_0$ such that

$$f(\boldsymbol{x}+\alpha \boldsymbol{y}_0)-f(\boldsymbol{x}) = \eta(\alpha \boldsymbol{y}_0)$$

for all $\boldsymbol{x} \in D$ and all $\alpha \in R$. Let $\boldsymbol{x}_0 \in D$ and $\xi \in \Gamma$. Then we have

$$\begin{aligned} \varphi(\xi) &= \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) \\ &\geqq \sup_{\alpha \in R} (\xi(\boldsymbol{x}_0+\alpha \boldsymbol{y}_0) - f(\boldsymbol{x}_0+\alpha \boldsymbol{y}_0)) \\ &= \sup_{\alpha \in R} (\xi - \eta)\alpha \boldsymbol{y}_0 + \xi \boldsymbol{x}_0 - f(\boldsymbol{x}_0), \end{aligned}$$

which implies $(\xi-\eta)\boldsymbol{y}_0 = 0$. Hence $\xi \boldsymbol{y}_0 = \eta \boldsymbol{y}_0$ for all $\xi \in \Gamma$, that is $\boldsymbol{y}_0 \in F_1$. Thus (iv) has been proved. The last statement follows from

$$F_1^{\perp} = (\Gamma - \xi)^{\perp\perp} = (m(\Gamma)-\xi)^{\perp\perp} = m(\Gamma) - \xi, \quad \xi \in \Gamma.$$

We shall now make some further assumptions on the vector spaces E_1 and E_2, namely that they are normed spaces, that the topologies induced by the norms are admissible and that

$$\|\boldsymbol{x}\| = \sup_{\substack{\|\xi\| \leqq 1 \\ \xi \in E_2}} |\xi \boldsymbol{x}|, \quad \|\xi\| = \sup_{\substack{\|\boldsymbol{x}\| \leqq 1 \\ \boldsymbol{x} \in E_1}} |\xi \boldsymbol{x}|$$

for all $\boldsymbol{x} \in E_1$ and all $\xi \in E_2$. In fact, this means that E_1 and E_2 are both reflexive Banach spaces, and each space is the dual of the other one.

Let F_1 be a closed subspace of E_1. It is easily verified that the function

$$\|\boldsymbol{x}\|_{F_1} = \sup_{\substack{\|\xi\| \leqq 1 \\ \xi \in F_1^{\perp}}} |\xi \boldsymbol{x}|$$

is a semi-norm in E_1, and that it has the following properties:

(i) $|\xi \boldsymbol{x}| \leqq \|\xi\| \cdot \|\boldsymbol{x}\|_{F_1}$ for $\boldsymbol{x} \in E_1$, $\xi \in F_1^{\perp}$.

(ii) $\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_{F_1}$, $\boldsymbol{x} \in E_1$, if and only if $F_1 = \{\boldsymbol{o}\}$.

We note that $\|\boldsymbol{x}\|_{F_1} = \inf_{\boldsymbol{z} \in F_1} \|\boldsymbol{x} - \boldsymbol{z}\|$ (cf. [7] p. 282).

5.2. THEOREM. *Let (D, f) be a closed convex function in E_1, and (Γ, φ) its conjugate in E_2. Let F_1 be the linearity space of (D, f). Then a point $\xi_0 \in E_2$ is a relative interior point of Γ if and only if there exist an $\boldsymbol{x}_0 \in E_1$, a $\varrho \in R_+$ and a $\sigma \in R$ such that*

$$f(\boldsymbol{x}) \geqq \varrho \|\boldsymbol{x} - \boldsymbol{x}_0\|_{F_1} - \sigma + \xi_0 \boldsymbol{x}$$

for all $\boldsymbol{x} \in D$. In particular, ξ_0 is an interior point of Γ if and only if there exist an $\boldsymbol{x}_0 \in E_1$, a $\varrho \in R_+$ and a $\sigma \in R$ such that

$$f(\boldsymbol{x}) \geqq \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \| - \sigma + \xi_0 \boldsymbol{x}$$

for all $\boldsymbol{x} \in D$.

Proof. First, suppose that

$$f(\boldsymbol{x}) \geqq \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \|_{F_1} - \sigma + \xi_0 \boldsymbol{x}$$

for all $\boldsymbol{x} \in D$. Then

$$\sup_{\boldsymbol{x} \in D} (\xi_0 \boldsymbol{x} - f(\boldsymbol{x})) \leqq \sigma$$

that is $\xi_0 \in \Gamma$. For every $\xi \in m(\Gamma) = \xi_0 + F_1^{\perp}$ such that $\| \xi - \xi_0 \| \leqq \varrho$ we have

$$\begin{aligned}
\sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - f(\boldsymbol{x})) &\leqq \sup_{\boldsymbol{x} \in D} (\xi \boldsymbol{x} - \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \|_{F_1} + \sigma - \xi_0 \boldsymbol{x}) \\
&= \sup_{\boldsymbol{x} \in D} ((\xi - \xi_0)(\boldsymbol{x} - \boldsymbol{x}_0) - \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \|_{F_1}) + (\xi - \xi_0) \boldsymbol{x}_0 + \sigma \\
&\leqq \sup_{\boldsymbol{x} \in D} ((\| \xi - \xi_0 \| - \varrho) \| \boldsymbol{x} - \boldsymbol{x}_0 \|_{F_1}) + (\xi - \xi_0) \boldsymbol{x}_0 + \sigma \\
&\leqq (\xi - \xi_0) \boldsymbol{x}_0 + \sigma .
\end{aligned}$$

Hence $\xi \in \Gamma$, which proves that ξ_0 is a relative interior point of Γ.

If for all $\boldsymbol{x} \in D$

$$f(\boldsymbol{x}) \geqq \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \| - \sigma + \xi_0 \boldsymbol{x},$$

then $F_1 = \{\boldsymbol{o}\}$. For let $\boldsymbol{z}_0 \in D$ and $\boldsymbol{y}_0 \in F_1$. There exists a continuous linear function η on F_1 such that for all $\alpha \in R$

$$\begin{aligned}
f(\boldsymbol{z}_0 + \alpha \boldsymbol{y}_0) &= f(\boldsymbol{z}_0) + \eta(\alpha \boldsymbol{y}_0) \\
&\geqq \varrho \| \boldsymbol{z}_0 + \alpha \boldsymbol{y}_0 - \boldsymbol{x}_0 \| - \sigma + \xi_0 (\boldsymbol{z}_0 + \alpha \boldsymbol{y}_0) \\
&\geqq \varrho |\alpha| \cdot \| \boldsymbol{y}_0 \| - \varrho \| \boldsymbol{z}_0 - \boldsymbol{x}_0 \| - \sigma + \xi_0 \boldsymbol{z}_0 + \xi_0 (\alpha \boldsymbol{y}_0).
\end{aligned}$$

This implies $\boldsymbol{y}_0 = \boldsymbol{o}$, that is $F_1 = \{\boldsymbol{o}\}$. Hence $\| \boldsymbol{x} \|_{F_1} = \| \boldsymbol{x} \|$ for all $\boldsymbol{x} \in D$, and the proof above yields that ξ_0 is an interior point of Γ.

Next, let ξ_0 be a relative interior point of Γ. Then φ is continuous at ξ_0 (cf. 2.10). Consequently, there exist a $\varrho \in R_+$ and a $\sigma \in R$ such that

$$K = \{\xi \mid \xi \in m(\Gamma), \| \xi - \xi_0 \| \leqq \varrho\} \subseteq \Gamma$$

and

$$\varphi(\xi) \leqq \sigma \quad \text{for all } \xi \in K.$$

For every $\boldsymbol{x} \in D$ we then have

$$\begin{aligned} f(\boldsymbol{x}) &= \sup_{\xi \in \Gamma} (\xi \boldsymbol{x} - \varphi(\xi)) \geq \sup_{\xi \in K} (\xi \boldsymbol{x} - \varphi(\xi)) \\ &\geq \sup_{\xi \in K} \xi \boldsymbol{x} - \sigma = \sup_{\xi \in K} (\xi - \xi_0) \boldsymbol{x} - \sigma + \xi_0 \boldsymbol{x} \\ &= \varrho \| \boldsymbol{x} \|_{F_1} - \sigma + \xi_0 \boldsymbol{x}. \end{aligned}$$

Thus, we have an equality of the form desired, with $\boldsymbol{x}_0 = \boldsymbol{o}$.

If ξ_0 is an interior point of Γ, then $F_1 = \{\boldsymbol{o}\}$. Hence $\| \boldsymbol{x} \| = \| \boldsymbol{x} \|_{F_1}$, and we have

$$f(\boldsymbol{x}) \geq \varrho \| \boldsymbol{x} \| - \sigma + \xi_0 \boldsymbol{x}$$

for all $\boldsymbol{x} \in D$.

Now, the main theorem of this section follows immediately from theorem 5.2 and the dual statement:

5.3. Theorem. *Let $\mathfrak{C}_1$ denote the class of closed convex functions (D, f) in E_1 with the following two properties:*

(i) *D has a non-empty relative interior.*

(ii) *For some $\boldsymbol{x}_0 \in E_1$, $\xi_0 \in E_2$, $\varrho \in R_+$ and $\sigma \in R$ we have*

$$f(\boldsymbol{x}) \geq \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \|_{F_1} - \sigma + \xi_0 \boldsymbol{x} \quad \text{for} \quad \boldsymbol{x} \in D,$$

F_1 denoting the linearity space of (D, f).

The class of conjugates of the functions in $\mathfrak{C}_1$ is the analogously defined class $\mathfrak{C}_2$ in E_2, and conversely. If (D, f) in E_1 and (Γ, φ) in E_2 are closed convex functions with the property (i), *such that each function is the conjugate of the other one, then (D, f) is in $\mathfrak{C}_1$ and (Γ, φ) is in $\mathfrak{C}_2$.*

The same statement holds for the subclass $\mathfrak{D}_1$ of $\mathfrak{C}_1$ consisting of those closed convex functions (D, f) for which

(i) *D has a non-empty interior.*

(ii) *For some $\boldsymbol{x}_0 \in E_1$, $\xi_0 \in E_2$, $\varrho \in R_+$ and $\sigma \in R$ we have*

$$f(\boldsymbol{x}) \geq \varrho \| \boldsymbol{x} - \boldsymbol{x}_0 \| - \sigma + \xi_0 \boldsymbol{x} \quad \text{for} \quad \boldsymbol{x} \in D.$$

In that case $F_1 = \{\boldsymbol{o}\}$ for the functions involved.

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